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RELAXATION OSCILLATIONS OF A NONEQUILIBRIUM GAS FILLING A VARIABLE-AREA CHANNEL

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Relaxation oscillations of a perfect gas (periodic variations of the parameters of the medium, slow, very fast, nearly discontinuous) in a variable-area channel were considered in [1]. Such oscillations arise when the frequency of external perturbations (that are incident on the open end of a channel or are caused by a piston oscillating at the end of the channel) is close to that of possible natural oscillations of the gas in the channel. A distinctive feature of these oscillations is that periodic shock waves appear. In the literature this phenomenon is also called nonlinear near-resonance oscillations [2-7].

The study below includes the case of a nonequilibrium (relaxing) medium. The process of establishing thermodynamic equilibrium in a gas is also called relaxation, but it does not have the distinctive features of relaxation oscillations (portions of a rapid abrupt change) since it is smooth. The chosen nonequilibrium process in a gas was the establishment of thermodynamic equilibrium between translational and vibrational degrees of freedom of the gas molecules (vibrational relaxation). This study, therefore, considers relaxation oscillations of a gas undergoing vibrational relaxation. It is my hope that a tangle of complex-sounding terms will not arise and the notation crisis that this would entail could be overcome.

1. The unsteady, one-dimensional flow of a viscous, heat-conducting gas admitting excitation of vibrational degrees of freedom of the molecules is described by the system of equations

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} = -\rho u\frac{1}{S}\frac{dS}{dx}, \quad \rho\frac{du}{dt} + \frac{\partial p}{\partial x} = \left(\zeta + \frac{4\eta}{3}\right)\frac{\partial^2 u}{\partial x^2},$$
$$\rho\frac{dh}{dt} = \frac{dp}{dt} + \left(\zeta + \frac{4\eta}{3}\right)\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \varkappa\frac{\partial^2 T}{\partial x^2}, \quad \frac{de_k}{dt} = \frac{1}{\tau}(e_k^* - e_k). \tag{1.1}$$

Here p is the pressure; ρ is the density; u is the velocity; e_k is the energy of the vibrational degrees of freedom of the molecules; T is the translational temperature; $h = (\gamma/\gamma - 1)/(p/\rho) + e_k$ is the total gas enthalpy; τ is the vibrational relaxation time; S is the channel cross section; $\varkappa$ is the thermal conductivity; η and ζ are the viscosity coefficients (ζ is the second viscosity); x is the spatial coordinate; and t is time. The equilibrium values of the vibrational energy e_k^* and τ can be expressed by the formula [8]

$$e_k^* = \theta_k R/(\exp(\theta_k/T) - 1), \quad \tau = \exp(k_2 T^{-1/3})\frac{1}{k_1 p},$$

where θ_k is the characteristic vibrational temperature; R is the gas constant; and k_i are positive constants that characterize the physical properties of the gas. Specific values are given in [8].

In classical gas dynamics the viscosity and thermal conduction processes are disregarded because κ, η, ζ are small under ordinary conditions. Since the study carried out here includes shock waves in which viscous friction and thermal conduction are important, viscous terms are written into the system (1.1). Shock waves are introduced below not as jumps but as very rapid continuous changes in the gas-state parameters.

We consider a steady subsonic flow of a relaxing gas along a channel; the flow can be considered to be not viscous or heat-conducting. Suppose that this flow is perturbed by weak periodic waves, which enter the channel inlet or are excited by a piston oscillating at the end of the channel. We assume that the amplitude of these perturbations is characterized by the small quantity $\delta(\delta << 1)$. Under this action oscillations of small amplitude δ_1, depending on $\delta(\delta_1 = \delta^N)$, arise in the channel. The perturbed values of the flow parameters can be sought as a series in δ_1, e.g.,

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$$u = u^0 + u_1 + u_2 + \ldots$$

(the superscript 0 labels parameters of the unperturbed flow, u_1 is a quantity of the order of δ_1, u_2 is of the order of δ_1^2, etc.)

We reduce the system (1.1) to the form

$$\frac{d_\pm}{dt}u \pm \frac{1}{\rho a}\frac{d_\pm}{dt}p = \mp ua\frac{1}{S}\frac{dS}{dx} \mp \frac{\gamma - 1}{a\tau}(e_k^* - e_k) + \varepsilon\frac{1}{\rho}\frac{\partial^2 u}{\partial x^2} \pm$$
$$\pm\, \varepsilon\frac{\gamma - 1}{\rho a}\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \bar{\varkappa}\frac{\partial^2 T}{\partial x^2}\right], \tag{1.2}$$
$$T\frac{d_0}{dt}s = -\frac{1}{\tau}(e_k^* - e_k) + \varepsilon\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \bar{\varkappa}\frac{\partial^2 T}{\partial x^2}\right], \quad \frac{d_0}{dt}e_k = \frac{1}{\tau}(e_k^* - e_k).$$

Here s is the entropy;

$$\frac{d_\pm}{dt} = \frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}; \quad \frac{d_0}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}; \quad \varepsilon = \zeta + \frac{4}{3}\eta; \quad \bar{\varkappa} = \frac{\varkappa}{\varepsilon}.$$

The quantities $u \pm \int dp/\rho a$ are called Riemann invariants. As $\tau \to \infty$ (ordinary gas dynamics) instead of examining how the perturbations of the parameters of the gas flow in the channel behave, we can study the behavior of the perturbations of the Riemann invariants. In a nonequilibrium medium the Riemann invariants, J_1 and J_2 describing the flow of an ordinary (perfect) gas are supplemented with quantities characterizing nonequilibrium processes in the gas: change in entropy and vibrational relaxation. Correspondingly, an entropy wave and a relaxation wave are added to acoustic perturbations. Perturbations of this type are transported by gas particles and they propagate along streamlines.

We introduce the notation

$$J_1 = u + \int\frac{dp}{\rho a} - u^0 - \left(\int\frac{dp}{\rho a}\right)^0, \quad J_2 = u - \int\frac{dp}{\rho a} - u^0 + \left(\int\frac{dp}{\rho a}\right)^0.$$

The boundary conditions can be formulated as

$$x = 0, \ J_2 = KJ_1 + \delta\sin(\omega t),$$
$$x = L, \ J_2 = NJ_1 \tag{1.3}$$

(L is the channel length). Indeed, when the channel has an open end the usual boundary condition has the form $p = p^0$ (constant static pressure) or in Riemann invariants $J_2 - J_1 = 0$, in which case $N = K = 1$. The propagation of entropy and relaxation waves in the channel has no effect on the pressure and velocity perturbations, whereby if conditions for the pressure or velocity are set at the channel ends the reflection coefficients do not change from the values calculated for an ordinary (perfect) gas. In this case we can assume that there are not entropy-relaxation waves at all. As an example of the boundary conditions when an entropy wave is generated by acoustic waves we point out the existence of a shock wave at the channel inlet. The reflection coefficients for this case were calculated in [9].

The problem of correctly formulating the boundary conditions for the open end of a channel has been discussed in the literature (see, e.g., [10]). It has been ascertained that constant static pressure can be chosen as the boundary condition for low frequencies of gas oscillations and low velocities of the drifting flow. If the velocity of the drifting flow is high, then the condition of a constant total pressure $p + 1/2\rho u^2$ is a more accurate boundary condition. At high oscillation frequencies so-called corrections for the open end must be introduced. The reflection coefficients in this case are extremely cumbersome and a computer is needed to calculate them. Graphs and appropriate formulas are given in [10].

Writing the parameters of perturbed flow in the channel as an expansion in δ_1, from the system (1.2) in the first approximation in δ_1 we obtain

$$\frac{d_\mp}{dt}u_1 \pm \frac{1}{\rho^0 a^0}\frac{d_\mp}{dt}p_1 = \mp (ua)_1\frac{1}{S}\frac{dS}{dx} \mp \left(\frac{u \pm a}{\rho a}\right)_1\frac{dp^0}{dx} - (u \pm a)_1\frac{du^0}{dx}$$
$$\pm \frac{\gamma - 1}{a^0\tau^0}\left[\left(\frac{a_1}{a^0} + \frac{\tau_1}{\tau^0}\right)(e_k^* - e_k)^0 - (e_{k1}^* - e_{k1})\right], \tag{1.4}$$

$$\frac{d_0}{dt}s_1 = -\left(\frac{u^0}{T^0}T_1 + u_1\right)\frac{ds^0}{dx} + \frac{1}{T^0}\tau_1\frac{(e_k^* - e_k)^0}{\tau^{0^2}} - \frac{1}{T^0}\frac{e_{k1}^* - e_{k1}}{\tau^0}, \qquad \frac{d_0}{dt}e_{k1} = -\frac{1}{u^0}u_1\frac{(e_k^* - e_k)^0}{\tau^0} - \tau_1\frac{(e_k^* - e_k)^0}{\tau^{0^2}} + \frac{e_{k1}^* - e_{k1}}{\tau^0},$$

where

$$e_{k1}^* = \gamma\exp\frac{\theta_k}{T^0}\frac{T_1}{T^0}\left(\frac{e_k^{*0}}{a^0}\right)^2; \quad \tau_1 = -\tau^0\left(\frac{p_1}{p^0} + \frac{k_2}{3T^{0^{1/3}}}\frac{T_1}{T^0}\right).$$

For terms of the second order of smallness in δ_1 from (1.2) we have

$$\begin{aligned}
&\frac{d_\pm}{dt}u_2 \pm \frac{1}{\rho^0a^0}\frac{d_\pm}{dt}p_2 = \mp(ua)_2\frac{1}{S}\frac{dS}{dx} \mp \left(\frac{u \pm a}{\rho a}\right)_2\frac{dp^0}{dx} - (u \pm a)_2\frac{du^0}{dx}\\
&\pm \frac{\gamma - 1}{a^0\tau^0}\left[\left(\frac{a_2}{a^0} + \frac{\tau_2}{\tau^0}\right)(e_k^* - e_k)^0 - (e_{k2}^* - e_{k2})\right] \mp \frac{u^0 \pm a^0}{(\rho^0a^0)^2}\frac{dp^0}{dx}\rho_1a_1\\
&+ \frac{\varepsilon}{\delta_1}\frac{1}{\rho^0}\left(\frac{\partial^2}{\partial x^2}u_1 \pm \frac{\gamma - 1}{a^0}\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \overline{\varkappa}\frac{\partial^2 T}{\partial x^2}\right]_1 - \frac{\rho_1}{\rho^0}\frac{d^2u^0}{dx^2}\right)\\
&- (u \pm a)_1\frac{\partial}{\partial x}u_1 \mp u_1a_1\frac{1}{S}\frac{dS}{dx} \mp \left(\frac{1}{\rho a}\right)_1\left(\frac{d_\pm}{dt}p_1 + (u \pm a)_1\frac{d}{dx}p^0\right) \qquad (1.5)\\
&\mp \frac{1}{\rho^0a^0}(u \pm a)_1\frac{\partial}{\partial x}p_1 \pm \frac{\gamma - 1}{a^0\tau^0}(e_k^* - e_k)_1\left(\frac{a_1}{a^0} + \frac{\tau_1}{\tau^0}\right)\\
&\mp \frac{\gamma - 1}{(a^0\tau^0)^2}a_1\tau_1(e_k^* - e_k)^0 \pm \frac{\varepsilon}{\delta_1}(\gamma - 1)\left(\frac{1}{\rho a}\right)_1\left[\frac{d}{dx}\left(u^0\frac{du^0}{dx}\right) + \overline{\varkappa}\frac{d^2T^0}{dx^2}\right],
\end{aligned}$$

$$\begin{aligned}
&\frac{d_0}{dt}s_2 = -\left(\frac{u^0}{T^0}T_2 + u_2\right)\frac{ds^0}{dx} + \frac{1}{\tau^0T^0}\left(\tau_2\frac{(e_k^* - e_k)^0}{\tau^0} - e_{k2}^* + e_{k2}\right) - u_1\frac{\partial}{\partial x}s_1\\
&- \frac{(e_k^* - e_k)^0}{(\tau^0T^0)^2}\tau_1T_1 + \frac{1}{\tau^0T^0}(e_k^* - e_k)_1\left(\frac{T_1}{T^0} + \frac{\tau_1}{\tau^0}\right) + \frac{\varepsilon}{\delta_1}\frac{1}{\rho^0T^0}\left(\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \overline{\varkappa}\frac{\partial^2 T}{\partial x^2}\right]_1\right.\\
&\left.- \left(\frac{\rho_1}{\rho^0} + \frac{T_1}{T^0}\right)\left[\frac{d}{dx}\left(u^0\frac{du^0}{dx}\right) + \overline{\varkappa}\frac{d^2T^0}{dx^2}\right]\right), \qquad \frac{d_0}{dt}e_{k2} = -\frac{(e_k^* - e_k)^0}{\tau^0}\left(\frac{1}{u^0}u_2 + \frac{1}{\tau^0}\tau_2\right) + \frac{e_{k2}^* - e_{k2}}{\tau^0} - \tau_1\frac{e_{k1}^* - e_{k1}}{\tau^{0^2}}.
\end{aligned}$$

We introduce the notation $J_{1m} = u_m + p_m/(\rho a)^0$, $J_{2m} = u_m - p_m/(\rho a)^0$, $J_{3m} = s_m$, $J_{4m} = e_{km}$, $m = 1, 2$. We rewrite the systems (1.2) and (1.2) as

$$\begin{aligned}
&\frac{d}{dt}J_{11} = a_{11}J_{11} + a_{12}J_{21} + a_{14}J_{41}, \quad \frac{dx}{dt} = u^0 + a^0,\\
&\frac{d}{dt}J_{21} = a_{21}J_{11} + a_{22}J_{21} + a_{24}J_{41}, \quad \frac{dx}{dt} = u^0 - a^0,\\
&\frac{d}{dt}J_{31} = a_{31}J_{11} + a_{32}J_{21} + a_{34}J_{41}, \quad \frac{dx}{dt} = u^0, \qquad (1.6)\\
&\frac{d}{dt}J_{41} = a_{41}J_{11} + a_{42}J_{21} + a_{44}J_{41}, \quad \frac{dx}{dt} = u^0;
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}J_{12} = a_{11}J_{12} + a_{12}J_{22} + a_{14}J_{42} + a_{10}, \quad \frac{dx}{dt} = u^0 + a^0, \quad \frac{d}{dt}J_{22} = a_{21}J_{12} + a_{22}J_{22} + a_{24}J_{42} + a_{20}, \quad \frac{dx}{dt} = u^0 - a^0,\\
&\frac{d}{dt}J_{32} = a_{31}J_{12} + a_{32}J_{22} + a_{34}J_{42} + a_{30}, \quad \frac{dx}{dt} = u^0, \quad \frac{d}{dt}J_{42} = a_{41}J_{12} + a_{42}J_{22} + a_{44}J_{42} + a_{40}, \quad \frac{dx}{dt} = u^0 \qquad (1.7)
\end{aligned}$$

(the coefficients a_{ij} can be calculated by using the systems (1.4) and (1.5).

TABLE 1

Gas	Pr	γ
Argon, helium	0,9	1,666
Neon	1,3	1,666
Air	1,4	1,4
Hydrogen	20	1,4

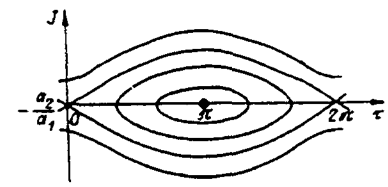

Fig. 1

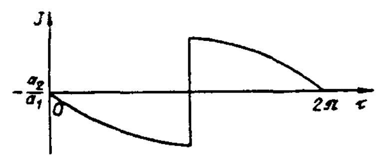

Fig. 2

The parameters of unperturbed steady flow in the channel are determined from the system of ordinary differential equations following from (1.1) when the dependence on time, viscous friction, and heat conduction are ignored:

$$u^0\frac{d\rho^0}{dx} + \rho^0\frac{du^0}{dx} = -\rho^0 u^0 \frac{1}{S}\frac{dS}{dx},\quad \rho^0 u^0\frac{du^0}{dx} + \frac{dp^0}{dx} = 0,$$
$$\rho^0 u^0\frac{dh^0}{dx} = u^0\frac{dp^0}{dx},\quad u^0\frac{d}{dx}e_k^0 = (e_k^* - e_k)^0/\tau^0,\quad h^0 = \frac{\gamma}{\gamma - 1}p^0/\rho^0 + e_k^0. \tag{1.8}$$

In the case of low-frequency oscillations the resulting equations can be solved by using the effective method of geometrical acoustics (or optics) [11] and looking for the solution as a series in inverse powers of ω ($\omega >> 1$ is the approximation of geometrical acoustics),

$$J_{lm} = \exp(i\omega(\sigma_l(x) - t))\sum_{n=0}^{\infty}\frac{j_{ln}(x)}{(i\omega)^n}.$$

When this approximation is used to solve the equations of the systems (1.6) and (1.7) the interplay of the Riemann invariants in these equations for each term of the expansion is a quantity that takes into account terms of higher order of smallness in $1/\omega$. Physically, this means that the time of interaction of waves propagating along the characteristics of various families is very short and in this sense this approximation is analogous to the "short wave" approximation [12].

The equations of the system (1.6) are solved in much the same way as in ordinary gas dynamics [1] (entropy perturbation waves and relaxation waves are not taken into account in view of the comment about the boundary conditions, see (1.3)). At the ends of the channel we have

$$J_{l1}(t, 0) = C_l(\omega t),\ J_{l1}(t, L) = C_l(\omega(t - \sigma_1(L)))\exp\left(\int_0^L \frac{a_{ll}}{u^0 \pm a^0}d\xi\right) + \frac{1}{\omega}\,[A_{lm}(L)F_m(\omega(t - \sigma_m(L)))$$
$$- A_{lm}(0)F_m(\omega(t - \sigma_l(L)))\,]\exp\left(\int_0^L \frac{a_{ll}}{u^0 \pm a^0}d\xi\right), \tag{1.9}$$

and in addition $F'_m = C_m$, $A_{lm} = \frac{a_{lm}}{u^0 \pm a^0}\exp\left(\int_0^x\left(\frac{a_{mm}}{u^0 \mp a^0} - \frac{a_{ll}}{u^0 \pm a^0}\right)d\xi\right)[(\sigma_l - \sigma_m)^{-1}]'$, $l \neq m$, $\sigma_l(x) = \int_0^x \frac{d\xi}{u^0 \pm a^0}$ where the prime denotes differentiation of the function with respect to its arguments and the superscript -1 denotes the inverse function;

$$J_{l2}(t,0)=0,\ J_{l2}(t,L)=\varepsilon b_{l1}(L)C_l''+b_{l2}(L)C_l'C_l,\qquad b_{lm}(x)=\int_0^x\frac{\hat{\alpha}_{lm}}{u^0\pm a^0}\exp\left(-\int_0^\xi\frac{a_{ll}}{u^0\pm a^0}d\xi\right)d\xi\exp\left(\int_0^x\frac{a_{ll}}{u^0\pm a^0}d\xi\right) \tag{1.10}$$

($\hat{\alpha}_{lm}$ are coefficients in the expansion $a_{l0}=\varepsilon\hat{\alpha}_{l1}C_l''+\hat{\alpha}_{l2}C_l'C_l$, l, m = 1, 2, the top sign corresponding to $l=1$ and the bottom sign, to $l=2$).

We substitute the solutions (1.9) and (1.10) into the boundary conditions (1.3) and upon eliminating C_2 we have

$$\begin{aligned}
&KC_1(\omega(t-\sigma_2(L)))+\frac{1}{\omega}\exp\left(-\int_0^L\frac{a_{22}}{u^0-a^0}d\xi\right)[\beta_{11}F_1(\omega(t-\sigma_1(L)))\\
&+\beta_{21}F_1(\omega(t-\sigma_2(L)))]+\varepsilon\exp\left(-\int_0^L\frac{a_{22}}{u^0-a^0}d\xi\right)b_{21}KC_1''(\omega(t-\sigma_2(L)))\\
&+\exp\left(-\int_0^L\frac{a_{22}}{u^0-a^0}d\xi\right)b_{22}K^2C_1'(\omega(t-\sigma_2(L)))C_1(\omega(t-\sigma_2(L)))\\
&=N\exp\left(-\int_0^L\frac{a_{22}}{u^0-a^0}d\xi\right)\Big\{C_1(\omega(t-\sigma_1(L)))\exp\left(\int_0^L\frac{a_{11}}{u^0+a^0}d\xi\right)\\
&+\frac{1}{\omega}K[\beta_{12}F_1(\omega(t-\sigma_2(L)))+\beta_{22}F_1(\omega(t-\sigma_1(L)))]+\varepsilon b_{11}C_1''(\omega(t-\sigma_1(L)))\\
&+b_{12}C_1'(\omega(t-\sigma_1(L)))C_1(\omega(t-\sigma_1(L)))\Big\}-\delta\sin(\omega(t-\sigma_2(L))),
\end{aligned} \tag{1.11}$$

$$\beta_{12}=A_{12}(L)\exp\left(\int_0^L\frac{a_{11}}{u^0+a^0}d\xi\right),\ \beta_{11}=A_{21}(L)\exp\left(\int_0^L\frac{a_{22}}{u^0-a^0}d\xi\right),$$

$$\beta_{22}=-A_{12}(0)\exp\left(\int_0^L\frac{a_{11}}{u^0+a^0}d\xi\right),\ \beta_{21}=-A_{21}(0)\exp\left(\int_0^L\frac{a_{22}}{u^0-a^0}d\xi\right).$$

We introduce the notation

$$T=\sigma_1(L)-\sigma_2(L),\ \eta=\omega(t-\sigma_1(L)),\qquad N_1=N\exp\left(\int_0^L\left(\frac{a_{11}}{u^0+a^0}-\frac{a_{22}}{u^0-a^0}\right)d\xi\right).$$

As shown in [1], $K=\dot{N}_1$ is a necessary condition for resonance. In the case of near-resonance oscillations $K=N_1-k$, where $k<<1$. Moreover, $\omega T=2\pi n+\Delta$, $\Delta<<1$, and we can use the expansion

$$C_1(\eta+\omega T)=C_1(\eta+2\pi n+\Delta)=C_1(\eta)+\Delta C_1'(\eta).$$

We take $C_1(\eta)=J(\eta)$ and recast the equation obtained from (1.11) by transformation in the form

$$\varepsilon\bar{a}_0J''+a_1JJ'+a_2J'+a_3J+a_4F=\delta\sin\eta, \tag{1.12}$$

where

$$\bar{a}_0=(b_{11}N-b_{21}K)\exp\left(-\int_0^L\frac{a_{22}}{u^0-a^0}d\xi\right);\quad a_1=(b_{12}N-b_{22}K^2)\exp\left(-\int_0^L\frac{a_{22}}{u^0-a^0}d\xi\right);$$

$$a_2=-K\Delta;\ a_3=-K+N_1=k;\qquad a_4=\exp\left(-\int_0^L\frac{a_{22}}{u^0-a^0}d\xi\right)\frac{1}{\omega}(KN(\beta_{12}+\beta_{22})-\beta_{11}-\beta_{21}).$$

A qualitative analysis of Eq. (1.12) with coefficients of general form was made in [1].

2. The coefficients of Eq. (1.12) in explicit form for a flow of ordinary perfect gas in a constant-area channel ($dS/dx\equiv 0$) with gas flowing in and out freely at the channel ends ($K=N=1$) are

$$\bar{a}_0=-\frac{L\omega^2}{2\delta\rho_0a_0^3}\left[\frac{(\gamma-1)\mathrm{M}_0-1-\frac{\bar{\varkappa}}{\gamma R}(\gamma-1)^2}{(1-\mathrm{M}_0)^3}-\frac{(\gamma-1)\mathrm{M}_0+1+\frac{\bar{\varkappa}}{\gamma R}(\gamma-1)^2}{(\mathrm{M}_0+1)^3}\right],$$

$$a_1=-\frac{L\omega(\gamma+1)}{4a_0^2}\left(\frac{1}{(\mathrm{M}_0-1)^2}-\frac{1}{(\mathrm{M}_0+1)^2}\right),\ a_2=-\Delta,\ a_4=a_3=0. \tag{2.1}$$

The parameters of an unperturbed constant flow in the channel are labeled with 0 subscripts.

The ratio of the viscosity and thermal conductivity is determined by the Prandtl number. In our case the expression

$$\frac{\bar{\varkappa}(\gamma - 1)}{\gamma R} = \frac{1}{\mathrm{Pr}}$$

can be considered as the reciprocal Prandtl number, calculated from the longitudinal viscosity $\varepsilon = (4/3)\eta + \zeta$. The values of Pr for various gases are shown in Table 1 [13]. Since the flow in the channel is subsonic everywhere, $M_0 < 1$, the coefficient $\bar{a}_0$ is positive, and a_1 is negative. We note that $a_1 \to 0$, $M_0 \to 0$.

We assume that $\Delta > 0$ (in general a change in the sign of a_2 is immaterial since when $a_2 \to -a_2$ the field of integral curves of Eq. (1.12) with $a_4 = 0$ goes over into itself upon the substitution $J \to -J(-\eta)$). We thus have the degenerate case $\bar{a}_0 > 0$, $a_1 < 0$, $a_2 < 0$, $a_3 = a_4 = 0$.

In this case the oscillations established in the channel do not contain shock waves. Indeed, the part of the solution that corresponds to the portion of rapid (nearly discontinuous) change can be approximated as a series in the small parameter ε [14]:

$$J = y_0(\eta^*) + \varepsilon y_1(\eta^*) + \ldots$$

($\eta^* = \eta/\varepsilon$). By substituting this expansion into (1.12) in the first approximation in ε we obtain

$$a_0 y_0'' + (a_1 y_0 + a_2) y_0' = 0. \tag{2.2}$$

The sign of the product $\bar{a}_0 a_1$ is determining in ascertaining the direction of the transition in the shocks. Suppose that y_{01} corresponds to the state before the shock and y_{02}, after the shock. It turns out that if $\bar{a}_0 a_1 < 0$ Eq. (2.2) has two solutions, corresponding to oscillations for which $y_{01} > y_{02}$ (expansion shocks). Indeed, Eq. (2.2) can be rewritten as

$$y_0'' = -\frac{1}{a_0}(a_1 y_0 + a_2) y_0'. \tag{2.3}$$

The signs of y_0'' and y_0' must be the same at the beginning of rapid fast oscillation (shock) and differ at the end. From (2.3) we see that the transition through the line $y_0 = -a_2/a_1$, corresponding to the shock, is accompanied by a change of sign of the coefficient in front of y_0' (but not the sign of y_0' itself). Accordingly, the sign of y_0'' changes. When $\bar{a}_0 a_1 < 0$, therefore, we can construct a solution containing an expansion shock by choosing $y_0' < 0$, $a_1 y_{01} + a_2 > 0$, $a_1 y_{02} + a_2 < 0$, i.e., $y_{01} > y_{02}$.

Solutions with expansion shocks are not realized in the problem under consideration. Indeed, the solutions of Eq. (1.12) represent the total change (caused by the arriving and reflected perturbation waves) in the Riemann invariant J_1 at the channel end $x = L$. If the channel end is open, then an arriving shock wave is reflected from it as an expansion shock. The total change in the invariant, however, is a shock wave: although the pressure jump at the channel exit is zero, gas flows out of the channel into the external space [15].

The oscillations established in the channel, therefore, are described by an equation that follows from (1.12) after the substitution $\bar{a}_0 = a_3 = a_4 = 0$ (the equality $\bar{a}_0 = 0$ formally signifies that there are no shocks in the channel):

$$(a_1 J + a_2) J' = \delta \sin \eta. \tag{2.4}$$

Equation (2.4) is integrated and makes it possible to obtain

$$J = \frac{1}{a_1}\left(-a_2 \pm \sqrt{a_2^2 - 2a_1(\delta \cos \eta + q^*)}\right). \tag{2.5}$$

(q^* is the constant of integration). The field of the integral curves (2.5) is given in Fig. 1.

If one end of the channel is closed by a solid wall while at the other end a piston generates weak oscillations, the unperturbed state of the gas in the channel can be assumed to be quiescence and the coefficients of the reflection of perturbations at the channel ends are $K = N = -1$. Then

$$\bar{a}_0 = -\frac{L\omega^2}{\delta\gamma\rho_0 a_0^3}\left[1 + \frac{\bar{\kappa}(\gamma-1)^2}{\gamma R}\right],\quad a_1 = -\frac{L\omega(\gamma+1)}{2a_0^2},$$
$$a_2 = \Delta,\quad a_4 = a_3 = 0. \tag{2.6}$$

In this case $\bar{a}_0 a_1 > 0$ and solutions corresponding to relaxation oscillations (containing shock waves) established in the channel can be constructed [1] (Fig. 2).

3. Let us consider the case when the channel cross section varies monotonically, causing the parameters of the flow in the channel to vary. Taking into account that K = N = 1, we can rewrite the coefficient a_3 as

$$a_3 = \exp\left(\int_0^L \frac{2 + (\gamma-1)M^{0^2}}{a_0(M^{0^2}-1)}\frac{du^o}{dx}dx\right) - 1.$$

As indicated above, the necessary condition for the existence of near-resonance oscillations in the channel is that k << 1 and, therefore, a_3 be small. This condition can be satisfied when the exponential term in the expression for a_3 is different from unity, which means that the argument of the exponent is a small quantity. This requirement can be satisfied if the derivative du/dx is itself a small quantity or changes sign so that the integral in the argument of the exponent is small.

The solution of the system (1.8) can be sought in the form

$$p^0 = p_0(1 - \bar{\varepsilon}\gamma p_{10}),\ \rho^0 = \rho_0(1 - \bar{\varepsilon}\rho_{10}),\ u^0 = u_0(1 + \bar{\varepsilon}u_{10}), \tag{3.1}$$

where $\bar{\varepsilon}$ is an auxiliary small parameter, e.g., for a channel with a monotonically varying cross section

$$\bar{\varepsilon} = \left|\ln\frac{S(L)}{S(0)}\right|. \tag{3.2}$$

Then

$$a_3 = \frac{\gamma-1}{a_0}\frac{M_0^2+2}{(M_0^2-1)}(u(L) - u(0)).$$

In an expanding channel u(L) < u(0), and so $a_3 > 0$. In a narrowing channel u(L) > u(0), and $a_3 < 0$.

For a_4 we have the expression

$$a_4 = \frac{1}{\omega}\frac{a_0 M_0}{1 - M_0^2}\left(\frac{1}{S(L)}\frac{dS}{dx}(L) - \frac{1}{S(0)}\frac{dS}{dx}(0)\right)\left(1 - \frac{\gamma-1}{2}M_0^2\right).$$

Since $M_0^2 < 1$, coefficient a_4 is positive if

$$\frac{1}{S(L)}\frac{dS}{dx}(L) > \frac{1}{S(0)}\frac{dS}{dx}(0). \tag{3.3}$$

On the other hand, a_4 is negative if

$$\frac{1}{S(L)}\frac{dS}{dx}(L) < \frac{1}{S(0)}\frac{dS}{dx}(0). \tag{3.4}$$

Examples of channels that satisfy the conditions (3.3) and (3.4) are given in Fig. 3.

Next we note that since the reflection coefficients at the channel boundaries are the same as before that the oscillations and the oscillations are assumed to be near-resonance ($\omega T = 2\pi n + \Delta$, $\Delta << 1$), the coefficient $a_2 = -K\Delta$ also retains the value calculated earlier (see (2.1)).

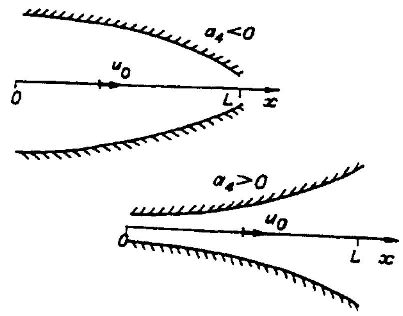

Fig. 3

When expansions of the type (3.1) are used the coefficients $\bar{a}_0$ and a_1 undergo a change of the order of $\bar{\varepsilon}$ in comparison with the case of constant uniform flow in the channel. Such additions do not change the sign of these coefficients and change their magnitude only slightly. These changes can be ignored and (2.1) can be used.

The effect of the variation of the channel cross section thus is taken into account by a change in the magnitude and sign of a_3 and a_4. Slight variation of the channel cross section has little effect on $\bar{a}_0$ and a_1 (changes their magnitude only slightly).

4. Let us consider the case of free flow of nonequilibrium gas from the channel. Suppose that the channel cross section is constant ($dS/dx \equiv 0$). Then

$$\begin{aligned}
a_{11} &= \frac{1}{2}\left[M^0(1 - M^0)\frac{\gamma + 1}{2} + \frac{(\gamma - 1)k_2}{3T^{0^{1/3}}}(1 - M^{0^2})\right]\frac{du^0}{dx} - E^0,\\
a_{22} &= \frac{1}{2}\left[-M^0(1 + M^0)\frac{\gamma + 1}{2} + \frac{(\gamma - 1)k_2}{3T^{0^{1/3}}}(1 - M^{0^2})\right]\frac{du^0}{dx} - E^0,\\
a_{12} &= -\frac{1}{2}\left[\left(2 + \frac{\gamma + 1}{2}M^0\right)(M^0 - 1) - \frac{(\gamma - 1)k_2}{3T^{0^{1/3}}}(1 - M^{0^2})\right]\frac{du^0}{dx} + E^0,\\
a_{21} &= -\frac{1}{2}\left[\left(-2 + \frac{\gamma + 1}{2}M^0\right)(M^0 + 1) - \frac{(\gamma - 1)k_2}{3T^{0^{1/3}}}(1 - M^{0^2})\right]\frac{du^0}{dx} + E^0.
\end{aligned} \tag{4.1}$$

Here

$$E^0 = \frac{\gamma}{2\tau^0}\left((\gamma - 1)\frac{e_k^{*0}}{a^{0^2}}\right)^2 \exp\frac{\theta_k}{T^0}.$$

The expressions obtained are used to calculate a_3:

$$a_3 = \exp\left(\int_0^L \frac{1}{a^0}\left\{\left(-\frac{(\gamma + 1)M^{0^2}}{1 - M^{0^2}} + \frac{(\gamma - 1)k_2}{3T^{0^{1/3}}}\right)\frac{du^0}{dx} + \frac{\gamma}{M^{0^2} - 1}\frac{1}{\tau^0}\left((\gamma - 1)\frac{e_k^{*0}}{a^{0^2}}\right)^2 \exp\frac{\theta_k}{T^0}\right\}dx\right) - 1.$$

As noted in Section 1, a necessary condition for near-resonance oscillations is that a_3 be small. This is possible if the exponential term in a_3 differs little from unity. The argument of the exponent, therefore, should be small. We consider the case when the smallness of the given argument is ensured by the smallness of the reciprocal relaxation time $1/\tau^0 << 1$.

This makes it possible to obtain a simple approximate solution of the system (1.8) by using the expansion for the unknown parameters of unperturbed flow of a gas in a channel in a form similar to that of (3.1):

$$p^0 = p_0(1 - \bar{\varepsilon}\gamma p_{10}),\ \rho^0 = \rho_0(1 - \bar{\varepsilon}\rho_{10}), \qquad u^0 = u_0(1 + \bar{\varepsilon}u_{10}),\ e_k^0 = a_0^2(\bar{e}_{k0} + \bar{\varepsilon}e_{k10}). \tag{4.2}$$

Here $\bar{\varepsilon}$ is an auxiliary small parameter (dimensionless reciprocal relaxation time calculated for $x = 0$):

$$\bar{\varepsilon} = \frac{1}{\bar{\tau}_0} = \frac{L}{\tau_0 u_0}. \tag{4.3}$$

Substituting (4.2) into (4.1) and integrating, we can find

$$a_3 = \frac{L}{M_0^2 - 1}\frac{\gamma - 1}{a_0^3\tau_0}\left\{\left(-\frac{(\gamma + 1)M_0^2}{1 - M_0^2} + \frac{(\gamma - 1)k_2}{3T_0^{1/3}}\right)(e_k^* - e_k)_0 + (\gamma - 1)\gamma\left(\frac{e_{k0}^*}{a_0}\right)^2 \exp\frac{\theta_k}{T_0}\right\}.$$

The resulting expression shows that in an equilibrium gas ($e^*_k = e_k$) $a_3 < 0$. For small M_0 fairly strong excitation of the vibrational degrees of freedom of the gas molecules results in a positive a_3. For large subsonic velocities the situation is the reverse and $a_3 < 0$.

We go on to calculate a_4. According to (4.1) and (4.2), the coefficients a_{km} are of the order of $\bar{\varepsilon}$ in the given approximation, and so we have

$$a_4 = \frac{1}{2\omega}\left[(a_{12}(L) - a_{12}(0))(1 - M_0) - (a_{21}(L) - a_{21}(0))(1 + M_0)\right]$$

and a_4 is a quantity of the second order of smallness in $\bar{\varepsilon}$. In the approximation under consideration we can set $a_4 = 0$.

In Section 3 we showed that when expansions of the form (4.2) are used $\bar{a}_0$ and a_1 undergo a change of the order of $\bar{\varepsilon}$ in comparison with the case of a constant uniform flow in the channel. Such additions do not change the sign of the given coefficients and change their magnitude only slightly. Disregarding these changes, we can use (2.1).

The effect of nonequilibrium processes (vibrational relaxation) is taken into account by the change in the magnitude and sign of a_3. A weak nonequilibrium state has little effect on the other coefficients.

In addition we consider forced oscillations of gas which is at rest in the channel and admits excitation of the vibrational degrees of freedom of the molecules. Suppose that one end of the channel is closed by a solid wall and at the other end a piston oscillates with a small amplitude. Here $K = N = -1$, the unperturbed state of the gas can be considered to be an equilibrium state with constant parameters. We have

$$a_{11} = a_{22} = -E_0,\ a_{12} = a_{21} = E_0,$$

$$a_3 = \frac{1}{1 - M_0^2}\frac{L}{\tau_0}\gamma(\gamma - 1)^2\left(\frac{e_{k0}^*}{a_0^2}\right)^2 \exp\frac{\theta_k}{T_0},\ a_4 = 0.$$

The coefficients $\bar{a}_0$, a_1, a_2 should be taken from (2.6).

All of the differences from ordinary gas dynamics in this case are in a_3. The effect of the term with this coefficient on the form of the solution of Eq. (1.12) may be decisive. Indeed, when the criterion

$$\left|\frac{\delta}{a_3}\right| < \left|\frac{a_2}{a_1}\right|$$

is satisfied solutions corresponding to discontinuous oscillations do not exist [1]. In ordinary gas dynamics (in the absence of relaxation $\tau \to \infty$) this criterion is not satisfied since $a_3 = 0$. The relaxation process in the gas as a result of the excitation of internal degrees of freedom can cause shock waves in the channel to disappear and smooth oscillations to be established.

5. The flow of a nonequilibrium gas in a variable-area channel can be studied by combining the results of Sections 3 and 4. Formally, the effect of weak variability of the channel area and the weak nonequilibrium, determined by small additions (of the order of $\bar{\varepsilon}$, introduced in accordance with Eqs. (3.2) and (4.3)), can be taken into account by simple addition because of their linearity.

We note that in cases when the product $\bar{a}_0 a_1$ is negative, a solution corresponding to oscillations with shock waves cannot be constructed [1]. In view of this we can conclude that free flows of a vibrationally relaxing gas in the channel are safe in regard to the formation of shock waves.

The formation of shock waves is known to be accelerated in a vibrationally excited gas because of the amplification of weak perturbations [12]. These processes (not related to resonances in the channels) are the source that initiates shock waves during flows of a nonequilibrium medium.

It is of interest to look for nonequilibrium media, during whose flow the conditions $\bar{a}_0 a_1 > 0$, $a_3 < 0$, since under these conditions relaxation oscillations (containing shock waves) arise and do not disappear even after the driving force ceases (i.e., are self-oscillations) [1].

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